Stability Analysis of an SII^*S Epidemic Model with Limited Treatment

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Abstract: There are many factors effect on the spread of infectious disease or control it, some of these factors is (treatment). And there are other factors that help the evolution of infectious diseases, for example (Negligence of the disease or mistake diagnosis of the disease). The main objective of this paper is to study the effect of those factors on the dynamical behavior of a SII^*S model. The impact of contact between of population and external sources of disease for example (air and other), on the dynamics of SII^*S epidemic model is investigated. The existence, uniqueness and boundedness of the solution of this model are investigated. The local and global dynamical behaviors of the model are studied. Finally, in order to confirm our obtained results and specify the effects of model's parameters on the dynamical behavior, numerical simulation of the SII^*S model is performed.

Keywords: Epidemic models, Stability, Treatment, External Source.

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1. Introduction

The mathematical models have become important tools in analyzing the spread and control of infectious diseases. The development of such models is aimed at both understand observed epidemiological patterns and predicting the consequences of the introduction of public health interventions to control the spread of diseases. Some diseases not confer immunity against the disease but other diseases confer immunity so recovered individuals gain immunity against disease. These types of disease can be modifications by *SI* and *SIS* where *S* susceptible and *I* infective respectively. Both epidemic models (*SI* and *SIS*) are one of the most basic and most important models in describing of many diseases. Therefore, it attached many authors attention and a number of papers have been published. For example Gao and Hethcote [1] considered an *SIS* model with a standard disease incidence and density-dependent demographics. Li and Ma [2] studied an *SIS* model with vaccination and temporary immunity. Kermack and Mckendeick [3] proposed a simple *SIS* model with infective immigrants. In recent years, many papers found treatment function for example, Li et al [4] proposed the *SIS* model with a limited resource for treatment. Shurowq k. Shafeeq [5] studied the effect of treatment, immigrants and vaccinated on the dynamic of *SIS* epidemic model. In this paper we proposed and studied a mathematical model consisting of SII^*S epidemic model with treatment, in which it is assumed that the disease transmitted by contact as well as external sources in the environment. The local as well as global stability analysis of this model is investigated.

2. The mathematical model

Consider a simple epidemiological model in which the total population (say $N(t)$) at time t is divided into two sub classes the susceptible individuals $S(t)$ and infected individuals $I(t)$. Such model can be represented as follows:

$$\begin{aligned} \frac{dS}{dt} &= \Lambda - \beta_1 SI - \mu S \\ \frac{dI}{dt} &= \beta_1 SI - \mu I \end{aligned} \qquad (1)$$

Here $\Lambda > 0$ is the recruitment rate of the population, $\mu > 0$ is the natural death rate of the population, $\beta_1 > 0$ is the infected rate (incidence rate) of susceptible individuals due to directed contact with the infected individuals.

Now, since there are many infectious disease for example (The flue., tube rculosis and cholera), spread in the environment by different factors including insects, contact or other vectors, therefore, we assumed that the disease in the a above model will transmitted between the population individuals by contact as well as external source of disease in the environment with an external source incidence rate $\beta_\circ \geq 0$. Also it is assumed that the nature recovery rate from infected individuals returns to be susceptible class with a constant rate $\alpha \geq 0$ and $\psi > 0$ is the rate of infected individuals from disease I into new disease I^*. Finally $\theta > 0$, $\beta_2 > 0$, the disease related death from second disease and the infected rate by contact between the susceptible individuals and infected individuals of second disease respectively. Then if addition above assumption system (1) can be rewritten in the form:

$$\frac{dS}{dt} = \Lambda - (\beta_\circ + \beta_1 I + \beta_2 I^*)S - \mu S + \alpha I$$
$$\frac{dI}{dt} = (\beta_\circ + \beta_1 I)S - (\alpha + \mu + \psi)I \quad (2)$$
$$\frac{dI^*}{dt} = \beta_2 S I^* + \psi I - (\mu + \theta)I^*$$

Keeping the above in view, in order to study the effect of treatment on the system (2) let *T(I)* represented the treatment function which given by [4]:

$$T(I) = \begin{cases} rI^* & if \quad 0 < I^* \le I_\circ^* \\ k & if \quad I^* > I_\circ^* \end{cases} \quad (3)$$

Accordingly, the flow of disease in system (2) along with the above assumptions can be representing in the following block diagram:

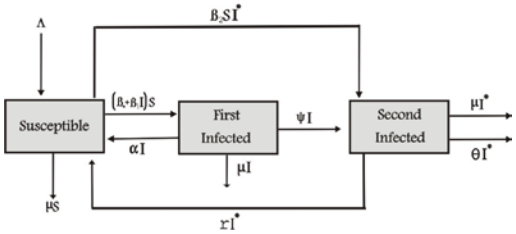

Figure (1): Block diagram of system (3).

Therefore, system (2) can be modified to:

$$\frac{dS}{dt} = \Lambda - (\beta_\circ + \beta_1 I + \beta_2 I^*)S - \mu S + \alpha I + T(I^*)$$
$$\frac{dI}{dt} = (\beta_\circ + \beta_1 I)S - (\alpha + \mu + \psi)I \quad (4)$$
$$\frac{dI^*}{dt} = \beta_2 S I^* + \psi I - (\mu + \theta)I^* - T(I^*)$$

her $k = rI_\circ^*$ this means that the treatment rate is proportional to the number of the infected individuals when the capacity of treatment is not reached, and otherwise takes the maximal capacity. Therefore at any point of time t the total number of population be comes $N(t) = S(t) + I(t) + I^*(t)$. Obviously, due to the biological meaning of the variables *S(t), I(t)* and $I^*(t)$, system (4) has the domain $R_+^3 = \{(S, I, I^*) \in R_+^3, S \ge 0, I \ge 0, I^* \ge 0\}$ which is positively in variant for system (4). Clearly, the interaction functions on the right hand said of system (4) are continuously differentiable. In fact they are Liptschizan function on R_+^3. Therefore, the solution of system (4) exits and unique. Further, all solutions of system (4) with non-negative initial conditions are uniformly bounded as shown in the following theorem.

Theorem (1): All the solutions of system (1), which are initiate in R_+^3, are uniformly bounded.

Proof: Let $(S(t), I(t), I^*(t))$ be any solution of the system (4) with non-negative initial conditions $(S(0), I(0), I^*(0))$. Since $N = S(t) + I(t) + I^*(t)$, then:

$$\frac{dN}{dt} = \frac{dS}{dt} + \frac{dI}{dt} + \frac{dI^*}{dt}$$

This gives

$$\frac{dN}{dt} = \Lambda - \mu(S + I + I^*) - \theta I^*$$

So, $\frac{dN}{dt} + \mu N \le \Lambda$

Now, by using Gronwall Lemma [6], it obtains that:

$$N(t) \le \frac{\Lambda}{\mu}(1 - e^{-\mu t}) + N(0)e^{-\mu t}$$

Therefore, $N(t) \le \frac{\Lambda}{\mu}$, as $t \to \infty$, hence all the solutions of system (4) that initiate in R_+^3 are confined in the reign:

$$\upsilon = \left\{(S, I, I^*) \in R_+^3 : N \le \frac{\Lambda}{\mu}\right\}$$

Which complete the proof. ■

3. Existence of equilibrium point of system (4)

The system (4) has at most three biologically feasible points, namely $E_i = (S_i, I_i, I_i^*), i = 0,1,2$. The existence conditions for each of these equilibrium points are discussed in the following:

1) If $I = 0$ and $I^* = 0$, then the system (4) has an equilibrium point called a disease free equilibrium point and denoted by $E_0 = (S_0, 0, 0)$ where:

$$S_0 = \frac{\Lambda}{\mu} \quad (5)$$

2) If $I^* = 0$, then the system (4) has an equilibrium point called a second disease free equilibrium point and denoted by $E_1 = (S_1, I_1, 0)$ where S_1 *and* I_1 represented the positive solution of the following set of equations:

$$\left.\begin{aligned} \Lambda - (\beta_\circ + \beta_1 I)S - \mu S + \alpha I = 0 \\ (\beta_\circ + \beta_1 I)S - (\alpha + \mu)I = 0 \end{aligned}\right\} \quad (6)$$

From equation (1) of above system we get:

$$S_1 = \frac{\Lambda + \alpha I_1}{\beta_\circ + \beta_1 I_1 + \mu} \quad (7a)$$

Substituting S_1 in equation (2) of system (6) we get:

$$I_1 = \frac{-D_2}{2D_1} - \frac{1}{2D_1}\sqrt{D_2^2 - 4D_1 D_3} \quad (7b)$$

her:

$$D_1 = -\beta_1 \mu$$
$$D_2 = \Lambda\beta_1 - \mu(\beta_\circ + \alpha + \mu)$$
$$D_3 = \beta_\circ \Lambda$$

Clearly, equation (7b) has a unique positive root by I_1 and then (E_2) exists uniquely in Int. R_+^3 if and only if $D_2 > 0$.

3) If $I \neq 0$ and $I^* \neq 0$ then the system (4) has an equilibrium point called endemic equilibrium point and denoted by $E_2 = (S_2, I_2, I_2^*)$ where S_2, I_2 and I_2^* represented the positive solution of the following set of equations in case $(0 < I^* < I_\circ^*)$ of equation (3) (treatment function):

$$\begin{aligned} &\Lambda - (\beta_\circ + \beta_1 I + \beta_2 I^*)S - \mu S + \alpha I + r I^* = 0 \\ &(\beta_\circ + \beta_1 I)S - (\alpha + \mu + \psi)I = 0 \\ &\beta_2 S I^* + \psi I - (\mu + \theta + r)I^* = 0 \end{aligned} \quad (8)$$

Straightforward computation to solve the above system of equations and from equation (2) and (3) of system (8) gives that:

$$S_2 = \frac{(\mu + \alpha + \psi)I_2}{\beta_\circ + \beta_1 I_2} \quad (9a)$$

$$I_2^* = \frac{-\psi I_2(\beta_\circ + \beta_1 I_2)}{\beta_2 I_2(\mu + \alpha + \psi) - (\mu + \theta + r)(\beta_\circ + \beta_1 I_2)} \quad (9b)$$

While, I_2^* positive root if and only if $\beta_2 I_2(\mu + \alpha + \psi) < (\mu + \theta + r)(\beta_\circ + \beta_1 I_2)$

Now, substituting S_2 and I_2^* in equation (1) of system (8) we get:

$$A_1 I_2^3 + A_2 I_2^2 + A_3 I_2 + A_4 = 0 \quad (10)$$

her:

$$\begin{aligned} A_1 = \beta_1 \{ &(\mu + \alpha + \psi)[\beta_1(\mu + \theta + r) + \alpha\beta_2] + \beta_1 \psi r - [\beta_2(\mu + \alpha + \psi)^2 \\ &+ \beta_2 \psi(\mu + \alpha + \psi) + \beta_1 \alpha(\mu + \theta + r)] \} \end{aligned}$$

$$\begin{aligned} A_2 = \{ &2\beta_\circ \beta_1 \psi r + (\mu + \alpha + \psi)[\Lambda\beta_1\beta_2 + \beta_\circ\beta_2\alpha + \beta_\circ\beta_1(\mu + \theta + r)] - \\ &[\beta_2(\mu + \alpha + \psi)[\beta_\circ\psi + (\beta_\circ + \mu)(\mu + \alpha + \psi)] + \beta_1(\mu + \theta + r) \\ &(\Lambda\beta_1 + 2\alpha\beta_\circ)] \} \end{aligned}$$

$$\begin{aligned} A_3 = \{ &(\mu + \alpha + \psi)\left[\Lambda\beta_\circ\beta_2 + (\mu + \theta + r)\left(\beta_\circ^2 + \beta_\circ\beta_1 + \beta_\circ\mu + \beta_1\mu\right)\right] \\ &+ \beta_\circ^2 \psi r - \beta_\circ(\mu + \theta + r)(2\beta_1 + \alpha\beta_\circ) \} \end{aligned}$$

$$A_4 = -\Lambda\beta_\circ^2(\mu + \theta + r) < 0$$

Clearly, equation (10) has a unique positive root by I_2 and then (E_2) exists uniquely in Int. R_+^3 if and only if $A_1 > 0$ then we have the following three cases:

Case (1): If the following conditions hold:

$$\left.\begin{aligned} A_2 > 0 \\ A_3 > 0 \end{aligned}\right\} \quad (11a)$$

Case (2): If the following conditions hold:

$$\left.\begin{aligned} A_2 < 0 \\ A_3 < 0 \end{aligned}\right\} \quad (11b)$$

Case (3): If the following conditions hold:

$$\left.\begin{aligned} A_2 > 0 \\ A_3 < 0 \end{aligned}\right\} \quad (11c)$$

4. Local stability analysis of system (4)

In this section, the local stability analysis of the equilibrium points $E_i, i = 0,1,2$ of the system (4) studied as shown in the following theorems.

Theorem (2): The disease free equilibrium point $E_0 = (S_0, 0, 0)$ of system (4) is locally asymptotically stable provided that:

$$\alpha < \beta_1 S_0 < \mu + \alpha + \psi \quad (12a)$$

$$r < \beta_2 S_0 < \mu + \theta + r \quad (12b)$$

$$(\beta_1 S_0 - \alpha)[\beta_0 + 2\mu + \alpha + \psi - \beta_1 S_0] > \psi(r - \beta_2 S_0) \quad (12c)$$

Proof: The Jacobian matrix of system (4) at (E_0) can be written as:

$$J(E_0) = \left| a_{ij} \right|_{3\times 3}$$

Where:

$$\begin{aligned} &a_{11} = -(\beta_\circ + \mu) \ ; \ a_{12} = \alpha - \beta_1 S_0 \ ; \ a_{13} = r - \beta_2 S_0 \\ &a_{21} = \beta_\circ \ ; \ a_{22} = \beta_1 S_0 - (\mu + \alpha + \psi) \ ; \ a_{23} = 0 \\ &a_{31} = 0 \ ; \ a_{32} = \psi \ ; \ a_{33} = \beta_2 S_0 - (\mu + \theta + r) \end{aligned}$$

Then the characteristic equation of $J(E_0)$ is given by:

$$\lambda^3 + \Omega_1\lambda^2 + \Omega_2\lambda + \Omega_3 = 0 \quad (13)$$

her:

$$\begin{aligned} \Omega_1 &= -[a_{11} + a_{22} + a_{33}] \\ &= (\beta_\circ + \mu) - (\beta_1 S_0 - (\mu + \alpha + \psi)) - (\beta_2 S_0 - (\mu + \theta + r)) \end{aligned}$$

$$\Omega_2 = a_{11}a_{22} - a_{12}a_{21} + a_{11}a_{33} + a_{22}a_{33}$$

$$\begin{aligned} \Omega_3 &= [a_{12}a_{21}a_{33} - a_{11}a_{22}a_{33} - a_{21}a_{32}a_{13}] \\ &= [\beta_\circ(\alpha - \beta_1 S_0)(\beta_2 S_0 - (\mu + \theta + r)) + (\beta_\circ + \mu) \\ &\quad (\beta_1 S_0 - (\mu + \alpha + \psi))(\beta_2 S_0 - (\mu + \theta + r)) - \beta_\circ\psi(r - \beta_2 S_0)] \end{aligned}$$

Further:

$$\begin{aligned} \Delta &= \Omega_1\Omega_2 - \Omega_3 \\ &= -a_{11}^2(a_{22} + a_{33}) - a_{22}^2(a_{11} + a_{33}) - a_{33}^2(a_{11} + a_{22}) \\ &\quad - a_{11}a_{22}a_{33} + a_{21}[a_{12}(a_{11} + a_{22}) + a_{32}a_{13}] \end{aligned}$$

$$\begin{aligned} &= -(\beta_\circ + \mu)^2[\beta_1 S_0 - (\mu + \alpha + \psi) + \beta_2 S_\circ - (\mu + \theta + r)] \\ &\quad - (\beta_1 S_0 - (\mu + \alpha + \psi))^2[-(\beta_\circ + \mu) + \beta_2 S_\circ - (\mu + \theta + r)] \\ &\quad - (\beta_2 S_\circ - (\mu + \theta + r))^2[-(\beta_\circ + \mu) + \beta_1 S_0 - (\mu + \alpha + \psi)] \\ &\quad + (\beta_\circ + \mu)(\beta_1 S_0 - (\mu + \alpha + \psi)(\beta_2 S_\circ - (\mu + \theta + r) + \beta_\circ \times \\ &\quad [(\alpha - \beta_1 S_0)(-(\beta_\circ + \mu) + \beta_1 S_0 - (\mu + \alpha + \psi)) + \psi(r - \beta_2 S_0)] \end{aligned}$$

Now, according to (Routh-Hurwitz) criterion [7], (E_0) will be locally asymptotically stable provided that $\Omega_1 > 0$; $\Omega_3 > 0$ and $\Delta = \Omega_1\Omega_2 - \Omega_3 > 0$. Clearly, $\Omega_i > 0, i = 1,3$ provided that conditions (12a)-(12b) hold. While, $\Delta = \Omega_1\Omega_2 - \Omega_3 > 0$,
Provided that conditions (12)-(a-c) hold.
Hence the proof is complete. ■

Theorem (3): The second disease free equilibrium point $E_1=(S_1,I_1,0)$ of system (4) is locally asymptotically stable if the following sufficient conditions are satisfied:

$$\mu > \max.\{2(\beta_1 S_1-\alpha-\psi),2(\beta_2 S_1-r)-\theta\} \quad (14)$$

Proof: The Jacobian matrix of system (4) at (E_1) that denoted by $J(E_1)$ can be written as:

$$J(E_1)=\lfloor b_{ij}\rfloor_{3\times 3}$$

Where:

$$b_{11}=-(\beta_\circ+\beta_1 I_1+\mu)\ ;\ b_{12}=\alpha-\beta_1 S_1\ ;\ b_{13}=r-\beta_2 S_1$$
$$b_{21}=\beta_\circ+\beta_1 I_1\ ;\ b_{22}=\beta_1 S_1-(\mu+\alpha+\psi)\ ;\ b_{23}=0$$
$$b_{31}=0\ ;\ b_{23}=\psi\ ;\ b_{33}=\beta_2 S_1-(\mu+\theta+r)$$

Now, according to *Gersgorin theorem* [8] if the following condition holds:

$$|b_{ii}|>\sum_{\substack{i=1\\ i\neq j}}^{3}|b_{ij}|$$

Then all eigenvalues of $J(E_1)$ exists in the region:

$$\wp=\bigcup\left\{U^*\in C:|U^*-b_{ii}|<\sum_{\substack{i=1\\ i\neq j}}^{3}|b_{ij}|\right\}$$

Therefore, according to the given condition (14) all the eigenvalues of $J(E_1)$ exists in the left half plane and hence, E_1 is locally asymptotically stable. ■

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Theorem(4): The endemic equilibrium point $E_2=(S_2,I_2,I_2^*)$ of system (4) is locally asymptotically stable if the following sufficient conditions are satisfied:

$$\mu > \max.\{2(\beta_1 S_2-\alpha-\psi),2(\beta_2 S_2-r)-\theta\} \quad (15)$$

Proof: The Jacobian matrix of system (4) at (E_2) that denoted by $J(E_2)$ can be written as:

$$J(E_2)=\lfloor c_{ij}\rfloor_{3\times 3}$$

Where:

$$c_{11}=-(\beta_\circ+\beta_1 I_2+\beta_2 I_2^*+\mu)\ ;\ c_{12}=\alpha-\beta_1 S_2\ ;\ c_{13}=r-\beta_2 S_2$$
$$c_{21}=\beta_\circ+\beta_1 I_2\ ;\ c_{22}=\beta_1 S_2-(\mu+\alpha+\psi)\ ;\ c_{23}=0$$
$$c_{31}=\beta_2 I_2^*\ ;\ c_{23}=\psi\ ;\ c_{33}=\beta_2 S_2-(\mu+\theta+r)$$

Now, according to *Gersgorin theorem* [8] if the following condition holds:

$$|c_{ii}|>\sum_{\substack{i=1\\ i\neq j}}^{3}|c_{ij}|$$

Then all eigenvalues of $J(E_2)$ exists in the region:

$$\varsigma=\bigcup\left\{U^*\in C:|U^*-c_{ii}|<\sum_{\substack{i=1\\ i\neq j}}^{3}|c_{ij}|\right\}$$

Therefore, according to the given condition (15) all the eigenvalues of $J(E_2)$ exists in the left half plane and hence, E_2 is locally asymptotically stable. ■

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5. Globally stability of system (4)

In this section, the global dynamics of system (4) is studied with the help of Lyapunov function as shown in the following theorems.

Theorem (5): Assume that, the disease free equilibrium point E_0 of system (4) is locally asymptotically stable. Then the basin of attraction of (E_0), say $B(E_0)\subset R_+^3$, it is globally asymptotically stable if satisfy the following condition:

$$(\beta_\circ+\beta_1 I+\beta_2 I^*)S<(\alpha I+rI^*) \quad (16)$$

Proof: Consider the following positive definite function:

$$V_1=\left(S-S_0-S_0\ln\frac{S}{S_0}\right)+I+I^*$$

Clearly, $V_1:R_+^3\to R$ is a continuously differentiable function such that $V_1(S_0,0,0)=0$, and $V_1(S,I,I^*)>0$, $\forall(S,I,I^*)\neq(S_0,0,0)$. Further we have:

$$\frac{dV_1}{dt}=\left(\frac{S-S_0}{S}\right)\frac{dS}{dt}+\frac{dI}{dt}+\frac{dI^*}{dt}$$

By simplifying this equation we get:

$$\frac{dV_1}{dt}=-\frac{\mu}{S}(S-S_0)^2+\left[\left(\beta_\circ+\beta_1 I+\beta_2 I^*\right)-\left(\frac{\alpha I+rI^*}{S}\right)\right]S_0 - \mu\left(I+I^*\right)-\theta I^*$$

Obviously, $\frac{dV_1}{dt}<0$, for every initial points and then V_1 is a Lyapunov function provided that condition (16) hold. Thus E_0 is globally asymptotically stable in the interior of $B(E_0)$, which means that $B(E_0)$ is the basin of attraction and that complete the proof. ■

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Theorem (6): Assume that, the second disease free equilibrium point E_1 of system (4) is locally asymptotically stable. Then the basin of attraction of (E_1), say $B(E_1)\subset R_+^3$, it is globally asymptotically stable if satisfy the following conditions:

$$\left(\frac{\beta_1 S_1 I-(\alpha I+\beta_\circ S+\beta_1 S I_1)}{SI}\right)^2<4\left(\frac{(\mu+\alpha+\psi)-\beta_1 S}{I}\right)\left(\frac{\beta_\circ+\mu+\beta_1 I}{S}\right) \quad (17a)$$

$$\left(\beta_2 S_1 I^*+\psi I\right)<\left(rS_1+(\mu+\theta)S\right)I^* \quad (17b)$$

Proof: Consider the following positive definite function:

$$V_2=\left(S-S_1-S_1\ln\frac{S}{S_1}\right)+\left(I-I_1-I_1\ln\frac{I}{I_1}\right)+I^*$$

Clearly, $V_2 : R_+^3 \to R$ is a continuously differentiable function such that $V_2(S_1, I_1, 0) = 0$, and $V_2(S, I, I^*) > 0$, $\forall (S, I, I^*) \neq (S_1, I_1, 0)$. Further we have:

$$\frac{dV_2}{dt} = \left(\frac{S - S_1}{S}\right)\frac{dS}{dt} + \left(\frac{I - I_1}{I}\right)\frac{dI}{dt} + \frac{dI^*}{dt}$$

By simplifying this equation we get:

$$\frac{dV_2}{dt} = -q_{11}(S - S_1)^2 - q_{12}(S - S_1)(I - I_1) - q_{22}(I - I_1)^2 - \left(\beta_2 - \frac{r}{S}\right)(S - S_1)I^* + \beta_2 S I^* + \psi I - (\mu + \theta + r)I^*$$

With:

$$q_{11} = \frac{\beta_\circ + \mu + \beta_1 I}{S} \; ; \; q_{22} = \frac{(\mu + \alpha + \psi) - \beta_1 S}{I} \; ;$$

$$q_{12} = \frac{\beta_1 S_1 I - (\alpha I + \beta_\circ S + \beta_1 S I_1)}{SI}$$

Therefore, according to condition (17a) it is obtaining that:

$$\frac{dV_2}{dt} \leq -\left[\sqrt{q_{11}}(S - S_1) + \sqrt{q_{22}}(I - I_1)\right]^2 + \beta_2 S_1 I^* + \psi I - (rS_1 + (\mu + \theta)S)I^*$$

Obviously, $\frac{dV_2}{dt} < 0$ for every initial points satisfying condition (17b) and then V_2 is a Lyapunov function provided that conditions (17a)-(17b) hold. Thus E_2 is globally asymptotically stable in the interior of $B(E_2)$, which means that $B(E_2)$ is the basin of attraction and that complete the proof. ■

Theorem (7): Let the endemic equilibrium point E_2 of system (4) is locally asymptotically stable. Then it is globally asymptotically stable provided that:

$$\max.\{\beta_2 S_2 - (\alpha + \psi), \beta_2 S_2 - (\theta + r)\} < \mu \quad (18a)$$

$$(\beta_\circ + \beta_1 I + \alpha - \beta_1 S_2)^2 < (\beta_\circ + \mu + \beta_1 I + \beta_2 I^*)(\mu + \alpha + \psi - \beta_1 S_2) \quad (18b)$$

$$(\beta_2 I^* + r - \beta_2 S_2)^2 < (\beta_\circ + \mu + \beta_1 I + \beta_2 I^*)(\mu + \theta + r - \beta_2 S_2) \quad (18c)$$

$$\psi^2 < (\mu + \alpha + \psi - \beta_1 S_2)(\mu + \theta + r - \beta_2 S_2) \quad (18d)$$

Proof: Consider the following positive definite function:

$$V_3 = \frac{(S - S_2)^2}{2} + \frac{(I - I_2)^2}{2} + \frac{(I^* - I_2^*)^2}{2}$$

Clearly, $V_3 : R_+^3 \to R$ is a continuously differentiable function such that $V_3(S_2, I_2, I_2^*) = 0$ and $V_3(S, I, I^*) > 0$, $\forall (S, I, I^*) \neq (S_2, I_2, I_2^*)$. Further, we have:

$$\frac{dV_3}{dt} = (S - S_2)\frac{dS}{dt} + (I - I_2)\frac{dI}{dt} + (I^* - I_2^*)\frac{dI^*}{dt}$$

By simplifying this equation we get:

$$\frac{dV_3}{dt} = -\frac{p_{11}}{2}(S - S_2)^2 + p_{12}(S - S_2)(I - I_2) - \frac{p_{22}}{2}(I - I_2)^2 - \frac{p_{11}}{2}(S - S_2)^2 + p_{13}(S - S_2)(I^* - I_2^*) - \frac{p_{33}}{2}(I^* - I_2^*)^2 - \frac{p_{22}}{2}(I - I_2)^2 + p_{23}(I - I_2)(I^* - I_2^*) - \frac{p_{33}}{2}(I^* - I_2^*)^2$$

With:

$$p_{11} = (\beta_\circ + \mu + \beta_1 I + \beta_2 I^*) \; ; \; p_{12} = (\beta_\circ + \beta_1 I + \alpha - \beta_1 S_2)$$

$$p_{22} = (\mu + \alpha + \psi) - \beta_1 S_2 \; ; \; p_{13} = (\beta_2 I^* + r - \beta_2 S_2)$$

$$p_{33} = (\mu + \theta + r) - \beta_2 S_2 \; ; \; p_{23} = \psi$$

Therefore, according to the conditions (18a)-(18d) we obtain that:

$$\frac{dV_3}{dt} \leq -\left[\sqrt{\frac{p_{11}}{2}}(S - S_2) - \sqrt{\frac{p_{22}}{2}}(I - I_2)\right]^2 - \left[\sqrt{\frac{p_{11}}{2}}(S - S_2) + \sqrt{\frac{p_{33}}{2}}(I^* - I_2^*)\right]^2 - \left[\sqrt{\frac{p_{22}}{2}}(I - I_2) + \sqrt{\frac{p_{33}}{2}}(I^* - I_2^*)\right]^2$$

Clearly, $\frac{dV_3}{dt} < 0$, and then V_3 is a Lyapunov function provided that the given conditions(18) (a-d) hold. Therefore, (E_2) is globally asymptotically stable. ■

6. Numerical analysis of systems (1)

In this section, the global dynamic of system (4) is studied numerically. The objectives of this study are confirming our obtained analytical results and understand the effects of contact, the external sources for disease and existence of treatment on the dynamic of SII^*S epidemic model. Consequently, the system (4) is solved numerically for different sets of initial conditions and for different sets of parameters. It is observed that, for the following set of hypothetical parameters that satisfies stability conditions (19a)-(19e) of endemic equilibrium point, system (4) has a globally asymptotically stable endemic equilibrium point as shown in following figure.

$$E = 500 \; ; \; \beta_\circ = 0.1 \; ; \; \beta_1 = 0.001 \; , \; \beta_2 = 0.001$$
$$\mu = 0.2 \; ; \; \alpha = 2 \; ; \; r = 2 \; ; \; \psi = 0.6 \; ; \; \theta = 0.4 \quad (19)$$

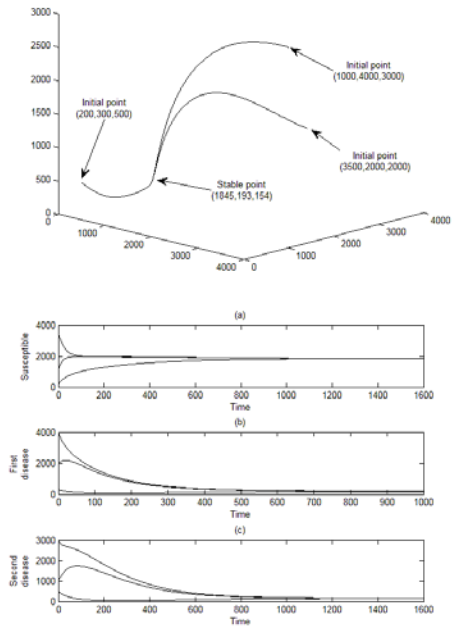

Figure 2- Phase plot and time series of system (4) starting from different initial points. *(a) Trajectories of S,* started at (3500, 1000, 200) *(b) trajectories of I,* started at (2000, 4000, 300) *(c) trajectories of* I^* started at (1000, 3000, 500).

Obviously, Figure (2) shows clearly the convergence of system (4) to the endemic equilibrium point $E_2 = (1845,193,154)$ asymptotically from three different initial points.

The effect of increasing the incidence rate of disease resulting from external sources on the dynamics of system (4) is studied by solving the system numerically for the parameters values $\beta_\circ = 0.001, 0.2, 0.4$ respectively, keeping other parameters fixed as given in equation (19), then the trajectories of system (4) are drawn in Figures (3a)- (3c) respectively and starting at *(3500, 2000, 1000).*

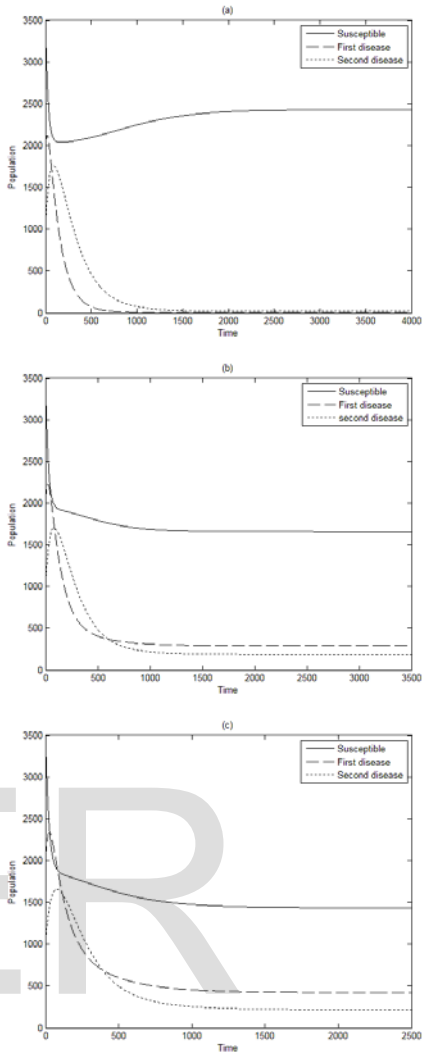

Figure 3- *Time series of the solution of system (4). (a) for* $\beta_\circ = 0.001$, *(b) for* $\beta_\circ = 0.2$, *(c) for* $\beta_\circ = 0.4$.

According to Figure (3), as the incidence rate of disease resulting by external sources increases *(through increasing* $\beta_\circ$*)*, then the trajectory of system (4) approaches asymptotically to the endemic equilibrium point. In fact as $\beta_\circ$ increases it is observed that the number of susceptible decrease and the number of infected in first disease individuals and infected in second disease individuals increases.

Similar results are obtained, as those shown in case of increasing $\beta_\circ$, in case of increasing the incidence rate of disease resulting by contact between susceptible and infected in first disease, that is means increasing β_1 and keeping other parameters fixed as given in (19).

The effect of increasing the incidence rate of disease resulting by contact between susceptible and infected in second disease on the dynamics of system (4) is studied by solving the system numerically for the parameters values $\beta_2 = 0.001, 0.004, 0.006$ respectively, keeping other

parameters fixed as given in equation (19). And then the trajectories of system (4) are drawn in Figures (4a)-(4c) respectively.

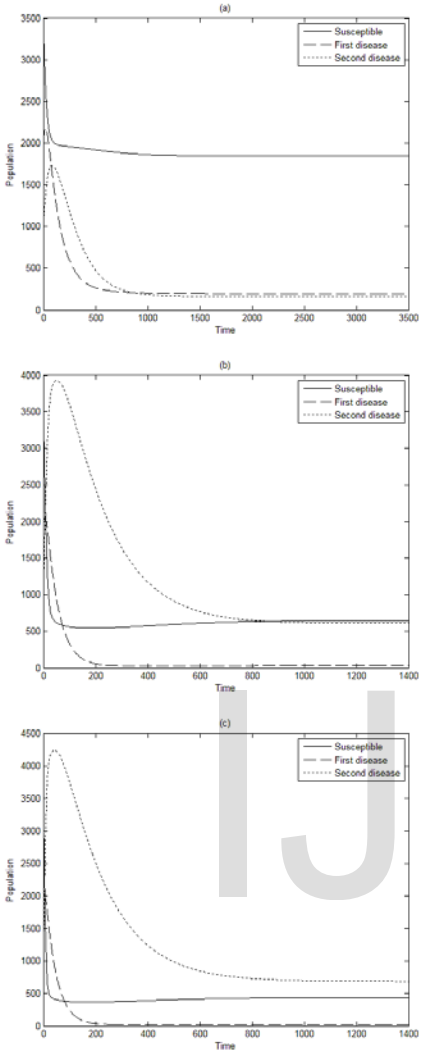

Figure 4- *Time series of the solution of system (4). (a) for* $\beta_2 = 0.001$, *(b) for* $\beta_2 = 0.004$, *(c) for* $\beta_2 = 0.006$. According to Figure (4), as the incidence rate of disease resulting by contact between susceptible individuals and infected in second disease increases, then the trajectory of system (4) still approaches asymptotically to the endemic equilibrium point. In fact as β_2 increases it is observed that the number of susceptible and infected in first disease individuals decrease and the number of infected in second disease individuals increases.

In the following, system (4) is solved numerically for the following values of natural recovery of first disease rates $\alpha = 1,\ 2.3\ , 4$, keeping other parameters fixed as given in equation (19), and then the trajectories of system (4) are drawn in Figures (5a)-(5c) respectively.

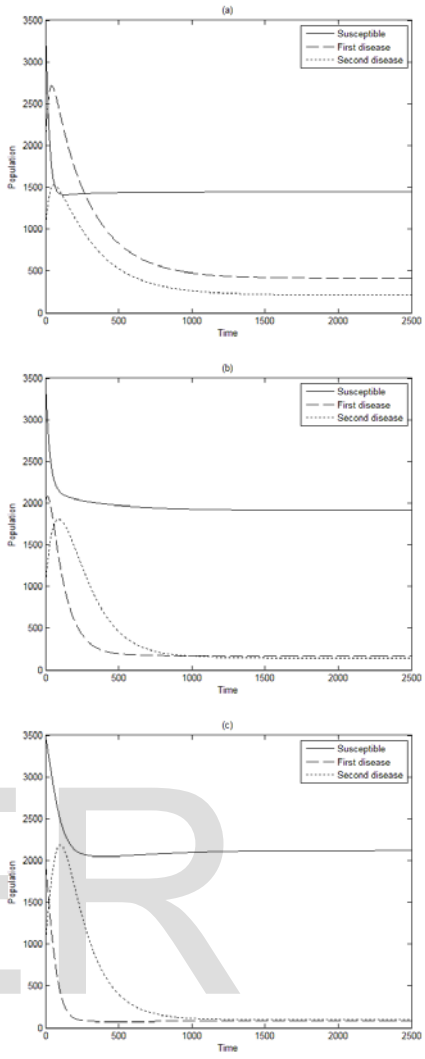

Figure 5- *Time series of the solution of system (4). (a) for* $\alpha = 1$, *(b) for* $\alpha = 2.3$, *(c) for* $\alpha = 4$.

According to Figure (5), as the natural recovery of first disease (α), then the trajectory of system (4) approaches asymptotically to the endemic equilibrium point. In fact as α increases it is observed that the number of susceptible individuals increase and the number of infected in first disease and infected in second disease individuals decreases.

Now, the effect of treatment rate on the dynamical behavior of system (4) is studied too. The system is solved numerically for different values of $r = 0.5,1,3$, keeping other parameters fixed as given in equation (19), and then the trajectories of system (4) are drawn in Figures (6a)-(6c) respectively.

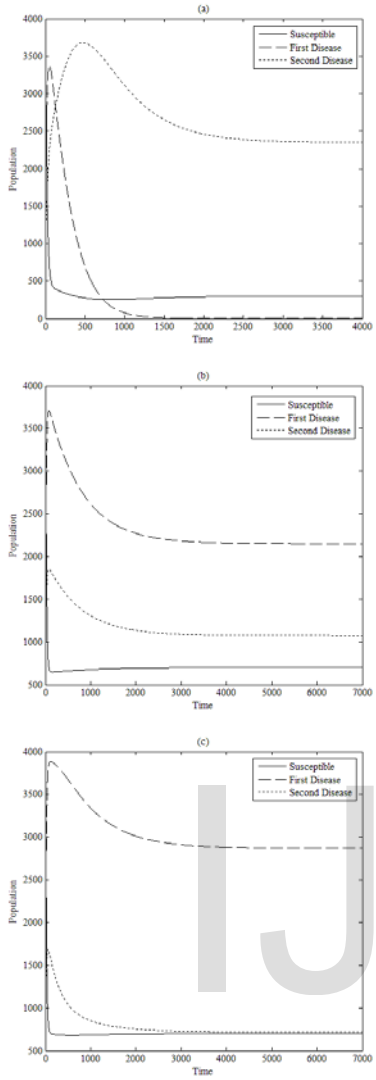

Figure 6- *Time series of the solution of system (4). (a) for* $r = 0.5$, *(b) for* $r = 1$, *(c) for* $r = 3$.

According to Figure (6), as the treatment (r), then the trajectory of system (4) approaches asymptotically to the endemic equilibrium point. In fact as r increases it is observed that the number of susceptible and infected in first disease individuals increase and the number of infected in second disease individuals decrease.

Similar results are obtained, as those shown in case of increasing r, in case of increasing the disease related death of second disease, that is means increasing θ and keeping other parameters fixed as given in (19). As shown in the following figures (7a)-(7c).

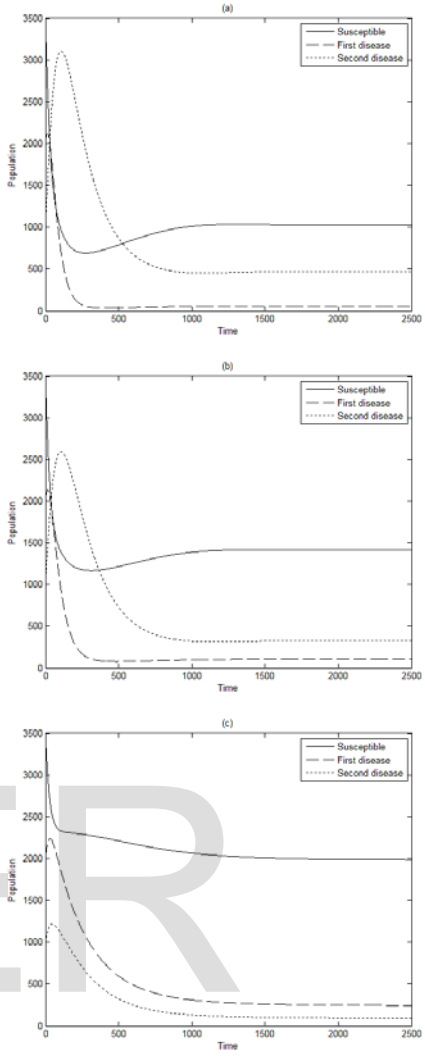

Figure 7- *Time series of the solution of system (4). (a) for* $\theta = 0.2$, *(b) for* $\theta = 0.5$, *(c) for* $\theta = 0.8$.

According to Figure (7), the disease related of second disease (θ), and then the trajectory of system (4) approaches asymptotically to the endemic equilibrium point. In fact as θ increases it is observed that the number of susceptible and infected in first disease individuals increase and the number of infected in second disease individuals decrease.

The effect of the natural death rate on the dynamics of system (4) is investigated numerically. It is observed that, increases the parameter μ and keeping other parameters fixed as in (19) then the trajectory of system (4) approaches asymptotically to the endemic equilibrium point as shown in Figures (8a)-(8b).

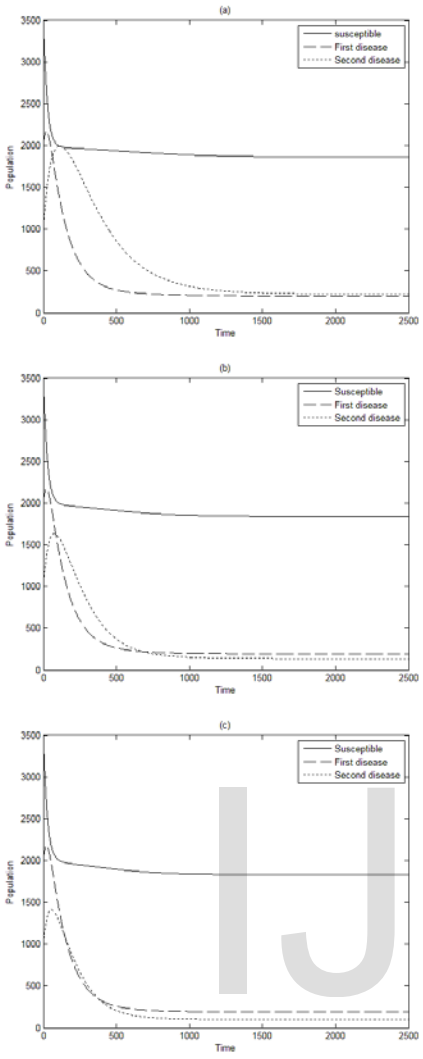

Figure 8- *Time series of the solution of system (4). (a) for* $\mu = 0.1$, *(b) for* $\mu = 0.2$, *(c) for* $\mu = 0.3$.

According to Figure (8), the natural death (μ), and then the trajectory of system (4) approach asymptotically to the endemic equilibrium point. In fact as μ increases it is observed that the number of susceptible individuals with the number of infected in first disease and second disease individuals decrease.

Finally, the effect of evolution rate of first disease and becomes to second disease that means increasing ψ, on the dynamical behavior of system (4) is studied. The system is solved numerically for different values of $\psi = 0.5, 0.7, 0.9$, keeping other parameters fixed as given in equation (20) and then the trajectory of system (4) as shown in Figures (9a)-(9c).

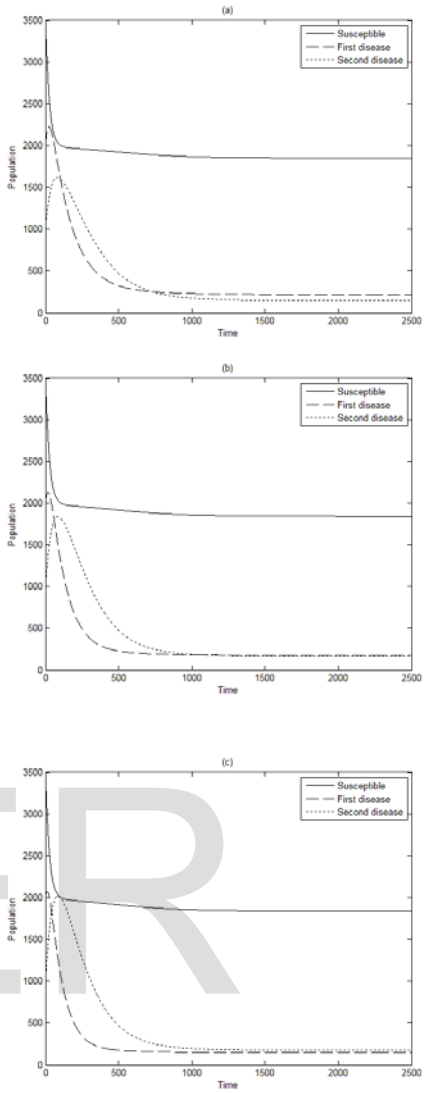

Figure 9- *Time series of the solution of system (4). (a) for* $\psi = 0.5$, *(b) for* $\psi = 0.7$, *(c) for* $\psi = 0.9$.

According to Figure (9), the evolution rate (ψ), and then the trajectory of system (4) approach asymptotically to the endemic equilibrium point. In fact as ψ increases it is observed that the numbers of susceptible individuals and second disease individuals increase with the number of infected in first disease individuals decrease.

7. Conclusion and discussion

In this paper, we proposed and analyzed an epidemiological model that described the dynamical behavior of an epidemic model, where the infectious disease transmitted directly from external sources as well as through contact between them. The model included fore non-linear autonomous differential equations that describe the dynamics of three different populations namely susceptible individuals (S), infected individuals for first disease (I) and infected individuals for second disease (evolution of first disease) (I^*). The boundedness of system (4) has been discussed. The conditions for existence,

stability for each equilibrium points are obtained. Further, it is observed that the disease free equilibrium point (E_0) exists when $I=0$ and locally stable if the conditions are hold (12) and it is globally stable if and only if the condition (16) holds. The second disease free equilibrium point (E_1) exists if ($D_2>0$) holds and locally stable if the conditions (14) are hold while it is globally stable if and only if the conditions (17a)-(17b) hold. The endemic equilibrium point (E_2) exists if $A_1>0$ and one of three conditions is hold (11a or 11b or 11c) and locally stable if the conditions (15) hold more than it is globally stable if and only if the conditions (18a)-(18d) hold. Finally, to understand the effect of varying each parameter on the global system (4) and confirm our above analytical results, the system (4) has been solved numerically for different sets of initial points and different sets of parameters given by equation (19), and the following observations are made:

1. The system (4) do not has periodic dynamic, instead it they approach either to the all equilibrium point.
2. As the incidence rate of disease (external incidence rate $(\beta_\circ)$ or contact incidence rate (β_1)) increase, the asymptotic behavior of the systems (4) approaching to endemic equilibrium point. In fact are $(\beta_i, i=\circ,1)$ increase it are observed that the number of (S) decrease and the number of $(I \text{ and } I^*)$ increase.
3. As the incidence rate of disease (contact incidence rate (β_2)) increase, the asymptotic behavior of the systems (4) approaching to endemic equilibrium point. In fact as (β_2) increase it is observed that the number of $(S \text{ and } I)$ decrease and the number of (I^*) increase.
4. As the natural recovery rate of first disease (α) the asymptotic behavior of the systems (4) approaching to endemic equilibrium point with increase it is observed that the number of (S) increase and the number of $(I \text{ and } I^*)$ decrease.
5. As the treatment rate (r) and the disease related death of second disease (θ) increase, the asymptotic behavior of the systems (4) approaching to endemic equilibrium point with increase it is observed that the number of $(S \text{ and } I)$ increase and the number of (I^*) decrease.
6. The increase in the natural death rate (μ), the asymptotic behavior of the systems (4) approaching to endemic equilibrium point with increase it is observed that the number of each the population $(S,\ I \text{ and } I^*)$ decrease.
7. As the evolution rate (ψ) increase, the asymptotic behavior of the systems (4) approaching to endemic equilibrium point with increase it is observed that the number of $(S \text{ and } I^*)$ increase and the number of (I) decrease.

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